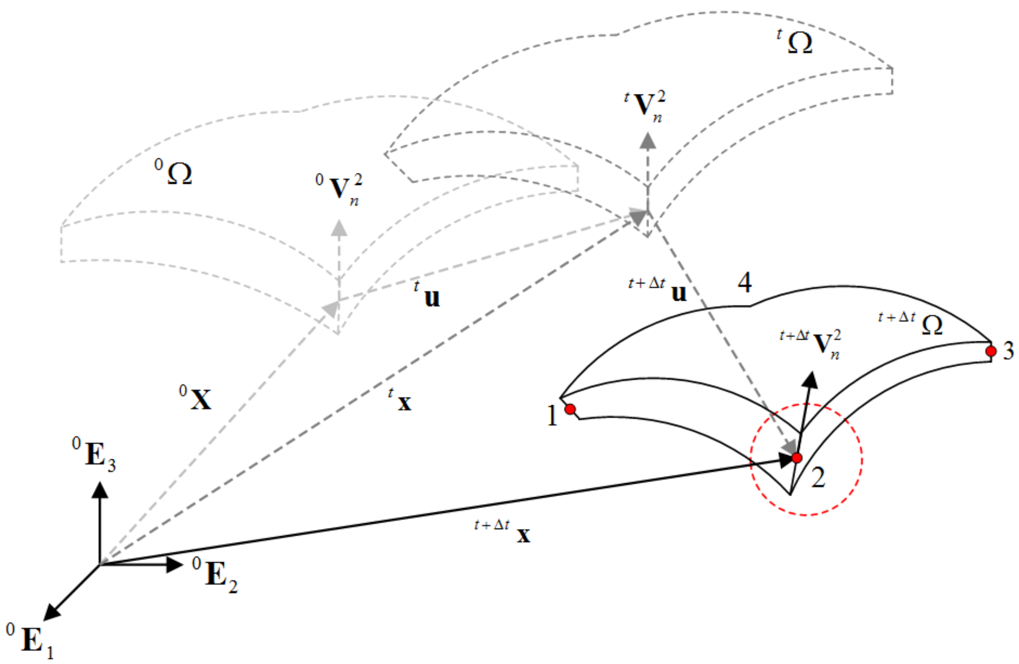
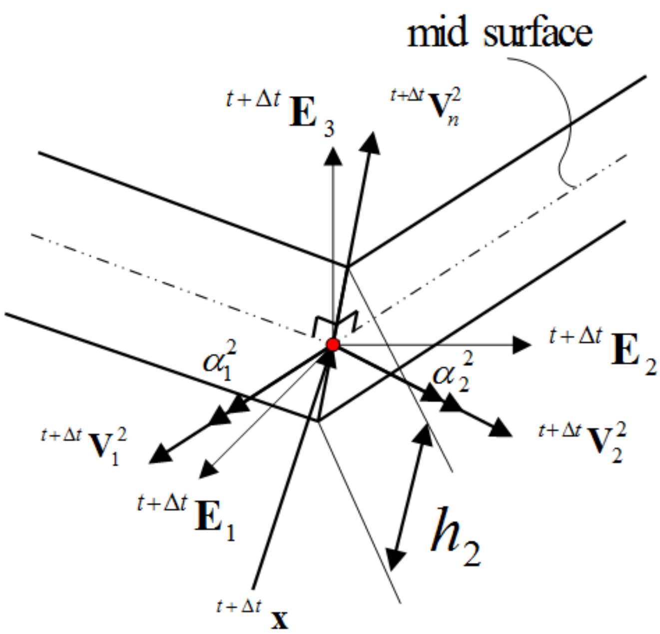
1. MITC shell element

- 3차원 솔리드 형상으로부터 쉘형상을 표현 (유한요소 정식화가 다른 쉘요소에 비해 간단)
- 쉘 이론을 사용하지 않고 3차원 응력, 변형률을 사용하여 쉘을 표현할 수 있다.
- 임의의 형상에 대한 두꺼운 쉘과 얇은 쉘 모두 적용 가능
- Locking을 방지하기 위해 횡방향 전단 변형률에 보간법 사용

2. Kinematics





Shell의 초기 위치 벡터는 다음과 같이 shape function으로 나타낼 수 있다.

$$ {}^{0}\mathbf{X} = \sum_{i=1}^{4} \phi_i \left(\xi^1, \xi^2\right) {}^{0}\mathbf{X}_i + \frac{\xi^3}{2} \sum_{i=1}^{4} h_i \phi_i \left(\xi^1, \xi^2\right) {}^{0}\mathbf{V}_n^i $$

마찬가지로 시간이 t, $t+\Delta t$ 일 때 위치벡터는 다음과 같다.

$$ {}^{t}\mathbf{x} = \sum_{i=1}^{4} \phi_i \left(\xi^1, \xi^2\right) {}^{t}\mathbf{x}_i + \frac{\xi^3}{2} \sum_{i=1}^{4} h_i \phi_i \left(\xi^1, \xi^2\right) {}^{t}\mathbf{V}_n^i $$

$$ {}^{t+\Delta t}\mathbf{x} = \sum_{i=1}^{4} \phi_i \left(\xi^1, \xi^2\right) {}^{t+\Delta t}\mathbf{x}_i + \frac{\xi^3}{2} \sum_{i=1}^{4} h_i \phi_i \left(\xi^1, \xi^2\right) {}^{t+\Delta t}\mathbf{V}_n^i $$

시간이 t 일 때와 $t+\Delta t$ 일 때 변위 벡터는 다음과 같이 계산 할 수 있다.

$$ \begin{aligned} {}^{t}\mathbf{u} &= {}^{t}\mathbf{x} - {}^{0}\mathbf{X} \\ &= \sum_{i=1}^{4} \phi_i \left(\xi^1, \xi^2\right) {}^{t}\mathbf{x}_i + \frac{\xi^3}{2} \sum_{i=1}^{4} h_i \phi_i \left(\xi^1, \xi^2\right) {}^{t}\mathbf{V}_n^i - \sum_{i=1}^{4} \phi_i \left(\xi^1, \xi^2\right) {}^{0}\mathbf{X}_i - \frac{\xi^3}{2} \sum_{i=1}^{4} h_i \phi_i \left(\xi^1, \xi^2\right) {}^{0}\mathbf{V}_n^i \\ &= \sum_{i=1}^{4} \phi_i \left({}^{t}\mathbf{x}_i - {}^{0}\mathbf{X}_i\right) + \frac{\xi^3}{2} \sum_{i=1}^{4} h_i \phi_i \left({}^{t}\mathbf{V}_n^i - {}^{0}\mathbf{V}_n^i\right) \end{aligned} $$

$$ \begin{aligned} {}^{t+\Delta t}\mathbf{u} &= {}^{t+\Delta t}\mathbf{x} - {}^{0}\mathbf{X} \\ &= \sum_{i=1}^{4} \phi_i \left(\xi^1, \xi^2\right) {}^{t+\Delta t}\mathbf{x}_i + \frac{\xi^3}{2} \sum_{i=1}^{4} h_i \phi_i \left(\xi^1, \xi^2\right) {}^{t+\Delta t}\mathbf{V}_n^i - \sum_{i=1}^{4} \phi_i \left(\xi^1, \xi^2\right) {}^{0}\mathbf{X}_i - \frac{\xi^3}{2} \sum_{i=1}^{4} h_i \phi_i \left(\xi^1, \xi^2\right) {}^{0}\mathbf{V}_n^i \\ &= \sum_{i=1}^{4} \phi_i \left({}^{t+\Delta t}\mathbf{x}_i - {}^{0}\mathbf{X}_i\right) + \frac{\xi^3}{2} \sum_{i=1}^{4} h_i \phi_i \left({}^{t+\Delta t}\mathbf{V}_n^i - {}^{0}\mathbf{V}_n^i\right) \end{aligned} $$

따라서 시간 t 와 $t+\Delta t$ 사이의 incremental displacement는 다음과 같이 나타낼 수 있다.

$$ \begin{aligned} \Delta^{t}\mathbf{u} &= {}^{t+\Delta t}\mathbf{u} - {}^{t}\mathbf{u} \\ &= \sum_{i=1}^{4} \phi_i \left({}^{t+\Delta t}\mathbf{x}_i - {}^{0}\mathbf{X}_i\right) + \frac{\xi^3}{2} \sum_{i=1}^{4} h_i \phi_i \left({}^{t+\Delta t}\mathbf{V}_n^i - {}^{0}\mathbf{V}_n^i\right) - \sum_{i=1}^{4} \phi_i \left({}^{t}\mathbf{x}_i - {}^{0}\mathbf{X}_i\right) - \frac{\xi^3}{2} \sum_{i=1}^{4} h_i \phi_i \left({}^{t}\mathbf{V}_n^i - {}^{0}\mathbf{V}_n^i\right) \\ &= \sum_{i=1}^{4} \phi_i \left({}^{t+\Delta t}\mathbf{x}_i - {}^{t}\mathbf{x}_i\right) + \frac{\xi^3}{2} \sum_{i=1}^{4} h_i \phi_i \left({}^{t+\Delta t}\mathbf{V}_n^i - {}^{t}\mathbf{V}_n^i\right) \\ &= \sum_{i=1}^{4} \phi_i \, \Delta^{t}\mathbf{u}_i + \frac{\xi^3}{2} \sum_{i=1}^{4} h_i \phi_i \left(-\alpha_1^i \, {}^{t}\mathbf{V}_2^i + \alpha_2^i \, {}^{t}\mathbf{V}_1^i\right) \\ &= \sum_{i=1}^{4} \phi_i \, \Delta^{t}\mathbf{u}_i + \frac{\xi^3}{2} \sum_{i=1}^{4} h_i \phi_i \Delta^{t}\mathbf{V}_n^i \end{aligned} $$

위치 벡터와 변위 벡터를 matrix form으로 나타내면

$$ {}^{0}\mathbf{X} = {}^{0}\mathbf{N}\,{}^{0}\mathbf{X}_n $$

$$ = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{bmatrix} \begin{bmatrix} {}^{0}\mathbf{X}_1 \\ {}^{0}\mathbf{X}_2 \\ {}^{0}\mathbf{X}_3 \\ {}^{0}\mathbf{X}_4 \end{bmatrix} + \begin{bmatrix} \frac{\xi^3}{2}h_1\phi_1 & \frac{\xi^3}{2}h_2\phi_2 & \frac{\xi^3}{2}h_3\phi_3 & \frac{\xi^3}{2}h_4\phi_4 \end{bmatrix} \begin{bmatrix} {}^{0}\mathbf{V}_n^1 \\ {}^{0}\mathbf{V}_n^2 \\ {}^{0}\mathbf{V}_n^3 \\ {}^{0}\mathbf{V}_n^4 \end{bmatrix} $$

$$ = \begin{bmatrix} \phi_1 & \frac{\xi^3}{2}h_1\phi_1 & \phi_2 & \frac{\xi^3}{2}h_2\phi_2 & \phi_3 & \frac{\xi^3}{2}h_3\phi_3 & \phi_4 & \frac{\xi^3}{2}h_4\phi_4 \end{bmatrix} \begin{bmatrix} {}^{0}\mathbf{X}_1 \\ {}^{0}\mathbf{V}_n^1 \\ {}^{0}\mathbf{X}_2 \\ {}^{0}\mathbf{V}_n^2 \\ {}^{0}\mathbf{X}_3 \\ {}^{0}\mathbf{V}_n^3 \\ {}^{0}\mathbf{X}_4 \\ {}^{0}\mathbf{V}_n^4 \end{bmatrix} $$

$$ {}^{t}\mathbf{x} = {}^{0}\mathbf{N}\,{}^{t}\mathbf{x}_n $$

$$ = \begin{bmatrix} \phi_1 & \frac{\xi^3}{2}h_1\phi_1 & \phi_2 & \frac{\xi^3}{2}h_2\phi_2 & \phi_3 & \frac{\xi^3}{2}h_3\phi_3 & \phi_4 & \frac{\xi^3}{2}h_4\phi_4 \end{bmatrix} \begin{bmatrix} {}^{t}\mathbf{x}_1 \\ {}^{t}\mathbf{V}_n^1 \\ {}^{t}\mathbf{x}_2 \\ {}^{t}\mathbf{V}_n^2 \\ {}^{t}\mathbf{x}_3 \\ {}^{t}\mathbf{V}_n^3 \\ {}^{t}\mathbf{x}_4 \\ {}^{t}\mathbf{V}_n^4 \end{bmatrix} $$

$$ {}^{t+\Delta t}\mathbf{x} = {}^{0}\mathbf{N}\,{}^{t+\Delta t}\mathbf{x}_n $$

$$ = \begin{bmatrix} \phi_1 & \frac{\xi^3}{2}h_1\phi_1 & \phi_2 & \frac{\xi^3}{2}h_2\phi_2 & \phi_3 & \frac{\xi^3}{2}h_3\phi_3 & \phi_4 & \frac{\xi^3}{2}h_4\phi_4 \end{bmatrix} \begin{bmatrix} {}^{t+\Delta t}\mathbf{x}_1 \\ {}^{t+\Delta t}\mathbf{V}_n^1 \\ {}^{t+\Delta t}\mathbf{x}_2 \\ {}^{t+\Delta t}\mathbf{V}_n^2 \\ {}^{t+\Delta t}\mathbf{x}_3 \\ {}^{t+\Delta t}\mathbf{V}_n^3 \\ {}^{t+\Delta t}\mathbf{x}_4 \\ {}^{t+\Delta t}\mathbf{V}_n^4 \end{bmatrix} $$

$$ \Delta^{t}\mathbf{x} = {}^{0}\mathbf{N}\,\Delta^{t}\mathbf{x}_n $$

$$ = \begin{bmatrix} \phi_1 & \frac{\xi^3}{2}h_1\phi_1 & \phi_2 & \frac{\xi^3}{2}h_2\phi_2 & \phi_3 & \frac{\xi^3}{2}h_3\phi_3 & \phi_4 & \frac{\xi^3}{2}h_4\phi_4 \end{bmatrix} \begin{bmatrix} \Delta^{t}\mathbf{x}_1 \\ \Delta^{t}\mathbf{V}_n^1 \\ \Delta^{t}\mathbf{x}_2 \\ \Delta^{t}\mathbf{V}_n^2 \\ \Delta^{t}\mathbf{x}_3 \\ \Delta^{t}\mathbf{V}_n^3 \\ \Delta^{t}\mathbf{x}_4 \\ \Delta^{t}\mathbf{V}_n^4 \end{bmatrix} $$

변위 벡터도 마찬가지로 나타낼 수 있다.

$$ {}^{t}\mathbf{u} = {}^{0}\mathbf{N}\left({}^{t}\mathrm{x}_n - {}^{0}\mathbf{X}_n\right) $$

$$ = \begin{bmatrix} \phi_1 & \frac{\xi^3}{2}h_1\phi_1 & \phi_2 & \frac{\xi^3}{2}h_2\phi_2 & \phi_3 & \frac{\xi^3}{2}h_3\phi_3 & \phi_4 & \frac{\xi^3}{2}h_4\phi_4 \end{bmatrix} \begin{bmatrix} {}^{t}\mathrm{x}_1 - {}^{0}\mathbf{X}_1 \\ {}^{t}\mathbf{V}_n^1 - {}^{0}\mathbf{V}_n^1 \\ {}^{t}\mathrm{x}_2 - {}^{0}\mathbf{X}_2 \\ {}^{t}\mathbf{V}_n^2 - {}^{0}\mathbf{V}_n^2 \\ {}^{t}\mathrm{x}_3 - {}^{0}\mathbf{X}_3 \\ {}^{t}\mathbf{V}_n^3 - {}^{0}\mathbf{V}_n^3 \\ {}^{t}\mathrm{x}_4 - {}^{0}\mathbf{X}_4 \\ {}^{t}\mathbf{V}_n^4 - {}^{0}\mathbf{V}_n^4 \end{bmatrix} $$

$$ {}^{t+\Delta t}\mathbf{u} $$

$$ = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{bmatrix} \begin{bmatrix} {}^{t+\Delta t}\mathrm{x}_1 - {}^{0}\mathbf{X}_1 \\ {}^{t+\Delta t}\mathrm{x}_2 - {}^{0}\mathbf{X}_2 \\ {}^{t+\Delta t}\mathrm{x}_3 - {}^{0}\mathbf{X}_3 \\ {}^{t+\Delta t}\mathrm{x}_4 - {}^{0}\mathbf{X}_4 \end{bmatrix} + \begin{bmatrix} \frac{\xi^3}{2}h_1\phi_1 & \frac{\xi^3}{2}h_2\phi_2 & \frac{\xi^3}{2}h_3\phi_3 & \frac{\xi^3}{2}h_4\phi_4 \end{bmatrix} \begin{bmatrix} {}^{t+\Delta t}\mathbf{V}_n^1 - {}^{0}\mathbf{V}_n^1 \\ {}^{t+\Delta t}\mathbf{V}_n^2 - {}^{0}\mathbf{V}_n^2 \\ {}^{t+\Delta t}\mathbf{V}_n^3 - {}^{0}\mathbf{V}_n^3 \\ {}^{t+\Delta t}\mathbf{V}_n^4 - {}^{0}\mathbf{V}_n^4 \end{bmatrix} $$

$$ = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{bmatrix} \begin{bmatrix} {}^{t+\Delta t}\mathbf{u}_1 \\ {}^{t+\Delta t}\mathbf{u}_2 \\ {}^{t+\Delta t}\mathbf{u}_3 \\ {}^{t+\Delta t}\mathbf{u}_4 \end{bmatrix} + \begin{bmatrix} \frac{\xi^3}{2}h_1\phi_1 & \frac{\xi^3}{2}h_2\phi_2 & \frac{\xi^3}{2}h_3\phi_3 & \frac{\xi^3}{2}h_4\phi_4 \end{bmatrix} \begin{bmatrix} \Delta^{t}\mathbf{V}_n^1 + \Delta^{0}\mathbf{V}_n^1 \\ \Delta^{t}\mathbf{V}_n^2 + \Delta^{0}\mathbf{V}_n^2 \\ \Delta^{t}\mathbf{V}_n^3 + \Delta^{0}\mathbf{V}_n^3 \\ \Delta^{t}\mathbf{V}_n^4 + \Delta^{0}\mathbf{V}_n^4 \end{bmatrix} $$

$$ \left({}^{t+\Delta t}\mathbf{V}_n^1 - {}^{0}\mathbf{V}_n^1 = {}^{t+\Delta t}\mathbf{V}_n^1 - {}^{t}\mathbf{V}_n^1 + {}^{t}\mathbf{V}_n^1 - {}^{0}\mathbf{V}_n^1 = \Delta^{t}\mathbf{V}_n^1 + \Delta^{0}\mathbf{V}_n^1\right) $$

$$ = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \phi_4 \end{bmatrix} \begin{bmatrix} {}^{t+\Delta t}\mathbf{u}_1 \\ {}^{t+\Delta t}\mathbf{u}_2 \\ {}^{t+\Delta t}\mathbf{u}_3 \\ {}^{t+\Delta t}\mathbf{u}_4 \end{bmatrix} + \begin{bmatrix} \frac{\xi^3}{2}h_1\phi_1 & \frac{\xi^3}{2}h_2\phi_2 & \frac{\xi^3}{2}h_3\phi_3 & \frac{\xi^3}{2}h_4\phi_4 \end{bmatrix} \begin{bmatrix} -\alpha_1^{1t}\mathbf{V}_2^1 + \alpha_2^{1t}\mathbf{V}_1^1 \\ -\alpha_1^{2t}\mathbf{V}_2^2 + \alpha_2^{2t}\mathbf{V}_1^2 \\ -\alpha_1^{3t}\mathbf{V}_2^3 + \alpha_2^{3t}\mathbf{V}_1^3 \\ -\alpha_1^{4t}\mathbf{V}_2^4 + \alpha_2^{4t}\mathbf{V}_1^4 \end{bmatrix} $$

$$ + \begin{bmatrix} \frac{\xi^3}{2}h_1\phi_1 & \frac{\xi^3}{2}h_2\phi_2 & \frac{\xi^3}{2}h_3\phi_3 & \frac{\xi^3}{2}h_4\phi_4 \end{bmatrix} \begin{bmatrix} \Delta^{0}\mathbf{V}_n^1 \\ \Delta^{0}\mathbf{V}_n^2 \\ \Delta^{0}\mathbf{V}_n^3 \\ \Delta^{0}\mathbf{V}_n^4 \end{bmatrix} $$

$$ \Rightarrow {}^{t+\Delta t}\mathbf{u} = {}^{t}\mathbf{N}\,{}^{t+\Delta t}\mathbf{u}_n + {}^{0}\tilde{\mathbf{N}}\Delta^{0}\tilde{\mathbf{X}}^n $$

$$ = \begin{bmatrix} \phi_1 & -\frac{\xi^3}{2}h_1\phi_1\mathbf{V}_2^1 & \frac{\xi^3}{2}h_1\phi_1\mathbf{V}_1^1 & \phi_2 & -\frac{\xi^3}{2}h_2\phi_2\mathbf{V}_2^2 & \frac{\xi^3}{2}h_2\phi_2\mathbf{V}_1^2 & \phi_3 & -\frac{\xi^3}{2}h_3\phi_3\mathbf{V}_2^3 & \frac{\xi^3}{2}h_3\phi_3\mathbf{V}_1^3 & \phi_4 & -\frac{\xi^3}{2}h_4\phi_4\mathbf{V}_2^4 & \frac{\xi^3}{2}h_4\phi_4\mathbf{V}_1^4 \end{bmatrix} \begin{bmatrix} {}^{t+\Delta t}\mathbf{u}_1 \\ \alpha_1^1 \\ \alpha_2^1 \\ {}^{t+\Delta t}\mathbf{u}_2 \\ \alpha_1^2 \\ \alpha_2^2 \\ {}^{t+\Delta t}\mathbf{u}_3 \\ \alpha_1^3 \\ \alpha_2^3 \\ {}^{t+\Delta t}\mathbf{u}_4 \\ \alpha_1^4 \\ \alpha_2^4 \end{bmatrix} $$

$$+\begin{bmatrix} \frac{\xi^3}{2}h_1\phi_1 & \frac{\xi^3}{2}h_2\phi_2 & \frac{\xi^3}{2}h_3\phi_3 & \frac{\xi^3}{2}h_4\phi_4 \end{bmatrix}\begin{bmatrix} \Delta^0\mathbf{V}_n^1 \\ \Delta^0\mathbf{V}_n^2 \\ \Delta^0\mathbf{V}_n^3 \\ \Delta^0\mathbf{V}_n^4 \end{bmatrix}$$

3. FE Formulation

현재 형상($t+\Delta t$)에서의 평형방정식은 다음과 같다.

$$\nabla_X \cdot {}^{t+\Delta t}\boldsymbol{\sigma} + \rho^{t+\Delta t}\mathbf{f} = \rho^{t+\Delta t}\ddot{\mathbf{u}}$$

가상일 원리를 적용하면

$$\int_V \delta^{t+\Delta t}\mathbf{u} \cdot \left(\nabla_X \cdot {}^{t+\Delta t}\boldsymbol{\sigma} + \rho^{t+\Delta t}\mathbf{f}\right) dV = \int_V \delta^{t+\Delta t}\mathbf{u} \cdot \rho^{t+\Delta t}\ddot{\mathbf{u}} dV$$

$$\Rightarrow \int_V \delta u_i \left(\frac{\partial \sigma_{ij}}{\partial x_j} + \rho f_i\right) dV = \int_V \delta u_i \cdot \rho \ddot{u}_i dV \qquad \left(\delta u_i \frac{\partial \sigma_{ij}}{\partial x_j} = \frac{\partial(\delta u_i \sigma_{ij})}{\partial x_j} - \frac{\partial \delta u_i}{\partial x_j}\sigma_{ij}\right)$$

$$\Rightarrow \int_V \left\{\frac{\partial(\delta u_i \sigma_{ij})}{\partial x_j} - \frac{\partial \delta u_i}{\partial x_j}\sigma_{ij} + \delta u_i \rho f_i\right\} dV = \int_V \delta u_i \cdot \rho \ddot{u}_i dV \qquad \rightarrow \textit{Gauss's divergence theorem}$$

$$\Rightarrow \int_{\partial V} \delta u_i \sigma_{ij} n_j dA + \int_V \left\{-\frac{\partial \delta u_i}{\partial x_j}\sigma_{ij} + \delta u_i \rho f_i\right\} dV = \int_V \delta u_i \cdot \rho \ddot{u}_i dV \quad \left(\textit{geometric B.C}\text{와 natural B.C}\right)$$

$$\Rightarrow \int_{\partial V_g} \delta u_i \sigma_{ij} n_j dA + \int_{\partial V_m} \delta u_i \bar{t}_i dA + \int_V \left\{-\frac{\partial \delta u_i}{\partial x_j}\sigma_{ij} + \delta u_i \rho f_i\right\} dV = \int_V \delta u_i \cdot \rho \ddot{u}_i dV$$

$$\left(\int_{\partial V_g} \delta u_i \sigma_{ij} n_j dA = 0, \text{ 기하학적 경계조건에서 가상변위가 0}\right)$$

$$\Rightarrow \int_{\partial V_m} \delta u_i \bar{t}_i dA + \int_V \left\{-\frac{\partial \delta u_i}{\partial x_j}\sigma_{ij} + \delta u_i \rho f_i\right\} dV = \int_V \delta u_i \cdot \rho \ddot{u}_i dV$$

$$\left(\frac{\partial \delta u_i}{\partial x_j}\sigma_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial x_j} + \frac{\partial \delta u_i}{\partial x_j}\right)\sigma_{ij} = \delta\varepsilon_{ij}\sigma_{ij}\right)$$

$$\Rightarrow \int_{\partial V_m} \delta u_i \bar{t}_i dA + \int_V \delta u_i \rho f_i dV = \int_V \delta u_i \cdot \rho \ddot{u}_i dV + \int_V \delta\varepsilon_{ij}\sigma_{ij} dV$$

비선형 해석을 위해 하중을 조금씩 증가시켜 물체의 변형을 순차적으로 구해 나가며 이러한 반복계산에 있어 기준이 되는 물체의 형상을 설정하는 방법에는 크게 Total Lagrange formulation과 Updated Lagrange formulation이 있다. Total Lagrange formulation은 변형과 관련된 변수들을 초기형상을 이용하여 정의하고 Updated Lagrange formulation은 변수들을 현재 변형된 형상으로 정의한다.

$$\int_V \delta u_i \cdot \rho \ddot{u}_i dV + \int_V \delta \varepsilon_{ij} \sigma_{ij} dV = \int_{\partial V_m} \delta u_i \bar{t}_i dA + \int_V \delta u_i \rho f_i dV$$

$$\left(\begin{array}{l} \int_V (\bullet) dV = \int_{V_0} (\bullet) \det(\mathbf{F}) dV_0 = \int_{V_0} (\bullet) J dV_0 \\ \int_{\partial V} (\bullet) \mathbf{n} dA = \int_{\partial V} (\bullet) J \mathbf{F}^{-T} \tilde{\mathbf{n}} dA \; (Nanson's \;\; formula) \\ \int_{V_0} \delta \varepsilon_{ij} \sigma_{ij} dV_0 = \int_{V_0} \delta F_{ij} P_{ij} dV_0 = \int_{V_0} \delta E_{ij} S_{ij} dV_0 \end{array} \right)$$

$$\Rightarrow \int_{V_0} \delta u_i \cdot \rho J \ddot{u}_i dV_0 + \int_{V_0} \delta E_{ij} S_{ij} dV_0 = \int_{\partial V_{0m}} \delta u_i \sigma_{ij} J \left[F^{-T} \right]_{kj} \tilde{n}_k dA_0 + \int_{V_0} \delta u_i \rho J f_i dV_0$$

$$\Rightarrow \int_{V_0} \delta u_i \cdot \rho_0 \ddot{u}_i dV_0 + \int_{V_0} \delta E_{ij} S_{ij} dV_0 = \int_{\partial V_{0m}} \delta u_i \sigma_{ij} J \left[F^{-T} \right]_{kj} \tilde{n}_k dA_0 + \int_{V_0} \delta u_i \rho_0 f_i dV_0$$

$$\Rightarrow \int_{V_0} \delta^{t+\Delta t}\mathbf{u} \cdot \rho_0{}^{t+\Delta t}\ddot{\mathbf{u}} dV_0 + \int_{V_0} \delta^{t+\Delta t}_{\;\;0}\mathbf{E} : {}^{t+\Delta t}_{\;\;0}\mathbf{S} dV_0 = \int_{\partial V_{0m}} \delta^{t+\Delta t}\mathbf{u} J^{t+\Delta t}\boldsymbol{\sigma} F^{-T\,t+\Delta t}\tilde{\mathbf{n}} dA_0 + \int_{V_0} \delta^{t+\Delta t}\mathbf{u} \rho_0{}^{t+\Delta t}\mathbf{f} dV_0$$

$$\left(J\boldsymbol{\sigma} F^{-T} = \mathbf{P} = \mathbf{F}\mathbf{S} \right)$$

위 식을 정리하면 다음과 같다.

$$\int_{V_0} \delta^{t+\Delta t}\mathbf{u} \cdot \rho_0{}^{t+\Delta t}\ddot{\mathbf{u}} dV_0 + \int_{V_0} \delta^{t+\Delta t}_{\;\;0}\mathbf{E} : {}^{t+\Delta t}_{\;\;0}\mathbf{S} dV_0 = \int_{\partial V_{0m}} \delta^{t+\Delta t}\mathbf{u}\, {}^{t+\Delta t}_{\;\;0}\mathbf{F}\, {}^{t+\Delta t}_{\;\;0}\mathbf{S}\, {}^{t+\Delta t}\tilde{\mathbf{n}} dA_0 + \int_{V_0} \delta^{t+\Delta t}\mathbf{u} \rho_0{}^{t+\Delta t}\mathbf{f} dV_0$$

$$\Leftrightarrow \int_{V_0} \delta^{t+\Delta t}\mathbf{u} \cdot \rho_0{}^{t+\Delta t}\ddot{\mathbf{u}} dV_0 + \int_{V_0} \delta^{t+\Delta t}_{\;\;0}\mathbf{F} : {}^{t+\Delta t}_{\;\;0}\mathbf{P} dV_0 = \int_{\partial V_{0m}} \delta^{t+\Delta t}\mathbf{u}\, {}^{t+\Delta t}_{\;\;0}\mathbf{P}\, {}^{t+\Delta t}\tilde{\mathbf{n}} dA_0 + \int_{V_0} \delta^{t+\Delta t}\mathbf{u} \rho_0{}^{t+\Delta t}\mathbf{f} dV_0$$

$\mathbf{E}$ 는 Green-Lagrange strain, $\mathbf{S}$ 는 2nd PK stress, $\mathbf{P}$ 는 1st PK stress, $\mathbf{F}$ 는 deformation gradient를 나타낸다. 여기에서는 Green-Lagrange strain과 2nd PK stress를 사용한다.

$$\int_{V_0} \delta^{t+\Delta t}\mathbf{u} \cdot \rho_0{}^{t+\Delta t}\ddot{\mathbf{u}} dV_0 + \int_{V_0} \delta^{t+\Delta t}_{\;\;0}\mathbf{E} : {}^{t+\Delta t}_{\;\;0}\mathbf{S} dV_0 = \int_{\partial V_{0m}} \delta^{t+\Delta t}\mathbf{u}\, {}^{t+\Delta t}_{\;\;0}\mathbf{F}\, {}^{t+\Delta t}_{\;\;0}\mathbf{S}\, {}^{t+\Delta t}\tilde{\mathbf{n}} dA_0 + \int_{V_0} \delta^{t+\Delta t}\mathbf{u} \rho_0{}^{t+\Delta t}\mathbf{f} dV_0$$

먼저 좌변을 살펴보면 첫 번째 항을 다음과 같이 정리 할 수 있다.

$$\int_{V_0} \delta^{t+\Delta t}\mathbf{u} \cdot \rho^{t+\Delta t}\ddot{\mathbf{u}} J dV_0 = \int_{V_0} \delta \left({}^{t}\mathbf{N}^{t+\Delta t}\mathbf{u}_n + {}^{0}\tilde{\mathbf{N}} \Delta {}^{0}\tilde{\mathbf{X}}_n \right) \cdot \rho^{t}\mathbf{N}^{t+\Delta t}\ddot{\mathbf{u}} J dV_0$$

$$= \int_{V_0} \delta \left({}^{t}\mathbf{N}^{t+\Delta t}\mathbf{u}_n \right) \cdot \rho \left({}^{t}\mathbf{N}^{t+\Delta t}\ddot{\mathbf{u}} \right) J dV_0$$

$$= \left[\delta^{t+\Delta t}\mathbf{u}_n \right]^T \int_{V_0} \rho \left[{}^{t}\mathbf{N} \right]^T \left[{}^{t}\mathbf{N} \right] J dV_0 \left[{}^{t+\Delta t}\ddot{\mathbf{u}}_n \right]$$

두 번째 항도 마찬가지로 다음과 같이 정리 할 수 있다. 먼저 Green-Lagrange strain은 다음과 같이 나타낼 수 있다.

$$
\begin{aligned}
{}^{t+\Delta t}_{0}\mathbf{E} &= \frac{1}{2}\left({}^{t+\Delta t}_{0}\mathbf{g}_i \cdot {}^{t+\Delta t}_{0}\mathbf{g}_j - {}_0\mathbf{G}_i \cdot {}_0\mathbf{G}_j \right)\left({}_0\mathbf{G}^i \otimes_0 \mathbf{G}^j \right) \\
&= \frac{1}{2}\left(\frac{\partial {}^{t+\Delta t}_{0}\mathbf{x}}{\partial \xi^i} \cdot \frac{\partial {}^{t+\Delta t}_{0}\mathbf{x}}{\partial \xi^j} - \frac{\partial {}^0\mathbf{X}}{\partial \xi^j} \cdot \frac{\partial {}^0\mathbf{X}}{\partial \xi^j} \right)\left({}_0\mathbf{G}^i \otimes_0 \mathbf{G}^j \right) \\
&= \frac{1}{2}\left(\frac{\partial\left({}^0\mathbf{X} + {}^{t+\Delta t}\mathbf{u} \right)}{\partial \xi^i} \cdot \frac{\partial\left({}^0\mathbf{X} + {}^{t+\Delta t}\mathbf{u} \right)}{\partial \xi^j} - \frac{\partial {}^0\mathbf{X}}{\partial \xi^j} \cdot \frac{\partial {}^0\mathbf{X}}{\partial \xi^j} \right)\left({}_0\mathbf{G}^i \otimes_0 \mathbf{G}^j \right) \\
&= \frac{1}{2}\left(\frac{\partial {}^0\mathbf{X}}{\partial \xi^j} \cdot \frac{\partial {}^0\mathbf{X}}{\partial \xi^j} + \frac{\partial {}^0\mathbf{X}}{\partial \xi^i} \cdot \frac{\partial {}^{t+\Delta t}\mathbf{u}}{\partial \xi^j} + \frac{\partial {}^{t+\Delta t}\mathbf{u}}{\partial \xi^i} \cdot \frac{\partial {}^0\mathbf{X}}{\partial \xi^j} + \frac{\partial {}^{t+\Delta t}\mathbf{u}}{\partial \xi^i} \cdot \frac{\partial {}^{t+\Delta t}\mathbf{u}}{\partial \xi^j} - \frac{\partial {}^0\mathbf{X}}{\partial \xi^j} \cdot \frac{\partial {}^0\mathbf{X}}{\partial \xi^j} \right)\left({}_0\mathbf{G}^i \otimes_0 \mathbf{G}^j \right) \\
&= \frac{1}{2}\left(\frac{\partial {}^0\mathbf{X}}{\partial \xi^i} \cdot \frac{\partial\left({}^t\mathbf{u} + \Delta {}^t\mathbf{u} \right)}{\partial \xi^j} + \frac{\partial\left({}^t\mathbf{u} + \Delta {}^t\mathbf{u} \right)}{\partial \xi^i} \cdot \frac{\partial {}^0\mathbf{X}}{\partial \xi^j} + \frac{\partial\left({}^t\mathbf{u} + \Delta {}^t\mathbf{u} \right)}{\partial \xi^i} \cdot \frac{\partial\left({}^t\mathbf{u} + \Delta {}^t\mathbf{u} \right)}{\partial \xi^j} \right)\left({}_0\mathbf{G}^i \otimes_0 \mathbf{G}^j \right)
\end{aligned}
$$

정리하면 Green-Lagrange strain을 $\Delta\mathbf{u}$에 대한 상수 term, 선형 term, 비선형 term으로 나눌 수 있다.

$$
{}^{t+\Delta t}_{0}\mathbf{E} = {}^{t+\Delta t}_{0}\mathbf{E}_0 + {}^{t+\Delta t}_{0}\mathbf{E}_C + {}^{t+\Delta t}_{0}\mathbf{E}_L
$$

$$
\left(
\begin{aligned}
{}^{t+\Delta t}_{0}\mathbf{E}_0 &= \frac{1}{2}\left(\frac{\partial {}^0\mathbf{X}}{\partial \xi^i} \cdot \frac{\partial {}^t\mathbf{u}}{\partial \xi^j} + \frac{\partial {}^t\mathbf{u}}{\partial \xi^i} \cdot \frac{\partial {}^0\mathbf{X}}{\partial \xi^j} + \frac{\partial {}^t\mathbf{u}}{\partial \xi^i} \cdot \frac{\partial {}^t\mathbf{u}}{\partial \xi^j} \right)\left({}_0\mathbf{G}^i \otimes_0 \mathbf{G}^j \right) \\
{}^{t+\Delta t}_{0}\mathbf{E}_C &= \frac{1}{2}\left(\frac{\partial {}^t\mathbf{u}}{\partial \xi^i} \cdot \frac{\partial \Delta {}^t\mathbf{u}}{\partial \xi^j} + \frac{\partial {}^0\mathbf{X}}{\partial \xi^i} \cdot \frac{\partial \Delta {}^t\mathbf{u}}{\partial \xi^j} + \frac{\partial \Delta {}^t\mathbf{u}}{\partial \xi^i} \cdot \frac{\partial {}^0\mathbf{X}}{\partial \xi^j} + \frac{\partial \Delta {}^t\mathbf{u}}{\partial \xi^i} \cdot \frac{\partial {}^t\mathbf{u}}{\partial \xi^j} \right)\left({}_0\mathbf{G}^i \otimes_0 \mathbf{G}^j \right) \\
{}^{t+\Delta t}_{0}\mathbf{E}_L &= \frac{1}{2}\left(\frac{\partial \Delta {}^t\mathbf{u}}{\partial \xi^i} \cdot \frac{\partial \Delta {}^t\mathbf{u}}{\partial \xi^j} \right)\left({}_0\mathbf{G}^i \otimes_0 \mathbf{G}^j \right)
\end{aligned}
\right)
$$

와 같다. 두 번째 항을 위의 표현으로 나타내면

$$
\begin{aligned}
&\int_{V_0} \delta {}^{t+\Delta t}_{0}\mathbf{E} : {}^{t+\Delta t}_{0}\mathbf{S} dV_0 = \int_{V_0} \delta {}^{t+\Delta t}_{0}\mathbf{E} : {}^{t+\Delta t}_{0}\mathbf{C} : {}^{t+\Delta t}_{0}\mathbf{E} dV_0 \\
&= \int_{V_0} \left(\underbrace{\delta {}^{t+\Delta t}_{0}\mathbf{E}_0}_{0} + \delta {}^{t+\Delta t}_{0}\mathbf{E}_C + \delta {}^{t+\Delta t}_{0}\mathbf{E}_L \right) : {}^{t+\Delta t}_{0}\mathbf{C} : \left({}^{t+\Delta t}_{0}\mathbf{E}_0 + {}^{t+\Delta t}_{0}\mathbf{E}_C + {}^{t+\Delta t}_{0}\mathbf{E}_L \right) dV_0 \\
&= \int_{V_0} \underbrace{\delta {}^{t+\Delta t}_{0}\mathbf{E}_C : {}^{t+\Delta t}_{0}\mathbf{C} : {}^{t+\Delta t}_{0}\mathbf{E}_0}_{\text{constant term}} dV_0 + \int_{V_0} \underbrace{\delta {}^{t+\Delta t}_{0}\mathbf{E}_C : {}^{t+\Delta t}_{0}\mathbf{C} : {}^{t+\Delta t}_{0}\mathbf{E}_C + \delta {}^{t+\Delta t}_{0}\mathbf{E}_L : {}^{t+\Delta t}_{0}\mathbf{C} : {}^{t+\Delta t}_{0}\mathbf{E}_0}_{\text{linear term}} dV_0 \\
&\quad + \int_{V_0} \underbrace{\delta {}^{t+\Delta t}_{0}\mathbf{E}_C : {}^{t+\Delta t}_{0}\mathbf{C} : {}^{t+\Delta t}_{0}\mathbf{E}_L + \delta {}^{t+\Delta t}_{0}\mathbf{E}_L : {}^{t+\Delta t}_{0}\mathbf{C} : {}^{t+\Delta t}_{0}\mathbf{E}_C + \delta {}^{t+\Delta t}_{0}\mathbf{E}_L : {}^{t+\Delta t}_{0}\mathbf{C} : {}^{t+\Delta t}_{0}\mathbf{E}_L}_{\text{high order term}} dV_0
\end{aligned}
$$

와 같다. 이후 선형화 시키고(High order term 무시) component형태로 나타내면 아래와 같다.

$$
\begin{aligned}
&\int_{V_0} \underbrace{\delta {}^{t+\Delta t}_{0}\mathbf{E}_C : {}^{t+\Delta t}_{0}\mathbf{C} : {}^{t+\Delta t}_{0}\mathbf{E}_0}_{\text{constant term}} dV_0 + \int_{V_0} \underbrace{\delta {}^{t+\Delta t}_{0}\mathbf{E}_C : {}^{t+\Delta t}_{0}\mathbf{C} : {}^{t+\Delta t}_{0}\mathbf{E}_C + \delta {}^{t+\Delta t}_{0}\mathbf{E}_L : {}^{t+\Delta t}_{0}\mathbf{C} : {}^{t+\Delta t}_{0}\mathbf{E}_0}_{\text{linear term}} dV_0 \\
&= \int_{V_0} \underbrace{\left[\delta {}^{t+\Delta t}_{0}\mathbf{E}_C \right]_{ij} \left[{}^{t+\Delta t}_{0}\mathbf{C} \right]^{ijkl} \left[{}^{t+\Delta t}_{0}\mathbf{E}_0 \right]_{kl}}_{\text{constant term}} dV_0 \\
&\quad + \int_{V_0} \underbrace{\left[\delta {}^{t+\Delta t}_{0}\mathbf{E}_C \right]_{ij} \left[{}^{t+\Delta t}_{0}\mathbf{C} \right]^{ijkl} \left[{}^{t+\Delta t}_{0}\mathbf{E}_C \right]_{kl} + \left[\delta {}^{t+\Delta t}_{0}\mathbf{E}_L \right]_{ij} \left[{}^{t+\Delta t}_{0}\mathbf{C} \right]^{ijkl} \left[{}^{t+\Delta t}_{0}\mathbf{E}_0 \right]_{kl}}_{\text{linear term}} dV_0 \\
&= \int_{V_0} \underbrace{\left[\delta {}^{t+\Delta t}_{0}\mathbf{E}_C \right]_{ij} \left[{}^{t+\Delta t}_{0}\mathbf{S}_0 \right]^{ij}}_{\text{constant term}} dV_0 + \int_{V_0} \underbrace{\left[\delta {}^{t+\Delta t}_{0}\mathbf{E}_C \right]_{ij} \left[{}^{t+\Delta t}_{0}\mathbf{S}_C \right]^{ij} + \left[\delta {}^{t+\Delta t}_{0}\mathbf{E}_L \right]_{ij} \left[{}^{t+\Delta t}_{0}\mathbf{S}_0 \right]^{ij}}_{\text{linear term}} dV_0
\end{aligned}
$$

Green-Lagrange strain의 변분을 구해보면 다음과 같다. 변분은 incremental displacement에만 적용되는데 이는 incremental displacement가 정의되지 않았기 때문이다. 따라서 $\delta^{t+\Delta t}\mathbf{u} = \delta\left({}^{t}\mathbf{u} + \Delta^{t}\mathbf{u}\right) = \delta\Delta^{t}\mathbf{u}$ 이고 $\delta^{t+\Delta t}_{\ \ 0}\mathbf{E}_0 = 0$ 이 성립한다.

$$\delta^{t+\Delta t}_{\ \ 0}\mathbf{E}_C = \frac{1}{2}\left(\frac{\partial^0\mathbf{X}}{\partial\xi^i}\cdot\frac{\partial\delta^{t+\Delta t}\mathbf{u}}{\partial\xi^j} + \frac{\partial\delta^{t+\Delta t}\mathbf{u}}{\partial\xi^i}\cdot\frac{\partial^0\mathbf{X}}{\partial\xi^j} + \frac{\partial\delta^{t+\Delta t}\mathbf{u}}{\partial\xi^i}\cdot\frac{\partial^{t+\Delta t}\mathbf{u}}{\partial\xi^j} + \frac{\partial^{t+\Delta t}\mathbf{u}}{\partial\xi^i}\cdot\frac{\partial\delta^{t+\Delta t}\mathbf{u}}{\partial\xi^j}\right)\left({}_0\mathbf{G}^i \otimes_0 \mathbf{G}^j\right)$$

$$\delta^{t+\Delta t}_{\ \ 0}\mathbf{E}_L = \frac{1}{2}\left(\frac{\partial\delta^{t+\Delta t}\mathbf{u}}{\partial\xi^i}\cdot\frac{\partial\Delta_0\mathbf{u}}{\partial\xi^j} + \frac{\partial\Delta_0\mathbf{u}}{\partial\xi^i}\cdot\frac{\partial\delta^{t+\Delta t}\mathbf{u}}{\partial\xi^j}\right)\left({}_0\mathbf{G}^i \otimes_0 \mathbf{G}^j\right)$$

Component form으로 나타내면

$$\begin{aligned}
\left[{}^{t+\Delta t}_{\ \ 0}\mathbf{E}_0\right]_{ij} &= \frac{1}{2}\left(\frac{\partial^0\mathbf{X}}{\partial\xi^i}\cdot\frac{\partial^t\mathbf{u}}{\partial\xi^j} + \frac{\partial^t\mathbf{u}}{\partial\xi^i}\cdot\frac{\partial^0\mathbf{X}}{\partial\xi^j} + \frac{\partial^t\mathbf{u}}{\partial\xi^i}\cdot\frac{\partial^t\mathbf{u}}{\partial\xi^j}\right) \\
&= \left[{}^0\mathbf{X}_n\right]^T \frac{1}{2}\left\{\left[\frac{\partial^0\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^0\mathbf{N}}{\partial\xi^j}\right] + \left[\frac{\partial^0\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^0\mathbf{N}}{\partial\xi^j}\right]\right\}\left[{}^t\mathbf{x}_n - {}^0\mathbf{X}_n\right] \\
&\quad + \frac{1}{2}\left[{}^t\mathbf{x}_n - {}^0\mathbf{X}_n\right]^T\left[\frac{\partial^0\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^0\mathbf{N}}{\partial\xi^j}\right]\left[{}^t\mathbf{x}_n - {}^0\mathbf{X}_n\right] \\
&= \left[{}^0\mathbf{X}_n\right]^T \frac{1}{2}\left\{\left[\frac{\partial^0\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^0\mathbf{N}}{\partial\xi^j}\right] + \left[\frac{\partial^0\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^0\mathbf{N}}{\partial\xi^j}\right]\right\}\left[{}^t\mathbf{x}_n - {}^0\mathbf{X}_n\right] \\
&\quad + \frac{1}{2}\left[{}^t\mathbf{x}_n - {}^0\mathbf{X}_n\right]^T \frac{1}{2}\left\{\left[\frac{\partial^0\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^0\mathbf{N}}{\partial\xi^j}\right] + \left[\frac{\partial^0\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^0\mathbf{N}}{\partial\xi^j}\right]\right\}\left[{}^t\mathbf{x}_n - {}^0\mathbf{X}_n\right] \\
&= \frac{1}{2}\left[{}^t\mathbf{x}_n + {}^0\mathbf{X}_n\right]^T \underbrace{\frac{1}{2}\left\{\left[\frac{\partial^0\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^0\mathbf{N}}{\partial\xi^j}\right] + \left[\frac{\partial^0\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^0\mathbf{N}}{\partial\xi^j}\right]\right\}}_{[\mathbf{e}]_{ij}}\left[{}^t\mathbf{x}_n - {}^0\mathbf{X}_n\right]
\end{aligned}$$

$$\begin{aligned}
\left[{}^{t+\Delta t}_{\ \ 0}\mathbf{E}_C\right]_{ij} &= \frac{1}{2}\left(\frac{\partial^t\mathbf{u}}{\partial\xi^i}\cdot\frac{\partial\Delta^t\mathbf{u}}{\partial\xi^j} + \frac{\partial^0\mathbf{X}}{\partial\xi^i}\cdot\frac{\partial\Delta^t\mathbf{u}}{\partial\xi^j} + \frac{\partial\Delta^t\mathbf{u}}{\partial\xi^i}\cdot\frac{\partial^0\mathbf{X}}{\partial\xi^j} + \frac{\partial\Delta^t\mathbf{u}}{\partial\xi^i}\cdot\frac{\partial^t\mathbf{u}}{\partial\xi^j}\right) \\
&= \left[{}^0\mathbf{X}_n\right]^T \frac{1}{2}\left\{\left[\frac{\partial^0\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^t\mathbf{N}}{\partial\xi^j}\right] + \left[\frac{\partial^t\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^0\mathbf{N}}{\partial\xi^j}\right]\right\}\left[\Delta^t\mathbf{u}_n\right] \\
&\quad + \left[{}^t\mathbf{x}_n - {}^0\mathbf{X}_n\right]^T \frac{1}{2}\left\{\left[\frac{\partial^t\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^0\mathbf{N}}{\partial\xi^j}\right] + \left[\frac{\partial^0\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^t\mathbf{N}}{\partial\xi^j}\right]\right\}\left[\Delta^t\mathbf{u}_n\right] \\
&= \left[{}^t\mathbf{x}_n\right]^T \underbrace{\frac{1}{2}\left\{\left[\frac{\partial^0\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^t\mathbf{N}}{\partial\xi^j}\right] + \left[\frac{\partial^t\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^0\mathbf{N}}{\partial\xi^j}\right]\right\}}_{[\mathbf{a}]_{ij}}\left[\Delta^t\mathbf{u}_n\right]
\end{aligned}$$

$$\begin{aligned}
\left[{}^{t+\Delta t}_{\ \ 0}\mathbf{E}_L\right]_{ij} &= \frac{1}{2}\left(\frac{\partial\Delta^t\mathbf{u}}{\partial\xi^i}\cdot\frac{\partial\Delta^t\mathbf{u}}{\partial\xi^j}\right) \\
&= \left[\Delta^t\mathbf{u}_n\right]^T \frac{1}{2}\left[\frac{\partial^t\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^t\mathbf{N}}{\partial\xi^j}\right]\left[\Delta^t\mathbf{u}_n\right]
\end{aligned}$$

$$\left[\delta^{t+\Delta t}{}_{0}\mathbf{E}_C\right]_{ij}=\frac{1}{2}\left(\frac{\partial^0\mathbf{X}}{\partial\xi^i}\cdot\frac{\partial\delta^{t+\Delta t}\mathbf{u}}{\partial\xi^j}+\frac{\partial\delta^{t+\Delta t}\mathbf{u}}{\partial\xi^i}\cdot\frac{\partial^0\mathbf{X}}{\partial\xi^j}+\frac{\partial\delta^{t+\Delta t}\mathbf{u}}{\partial\xi^i}\cdot\frac{\partial^{t+\Delta t}\mathbf{u}}{\partial\xi^j}+\frac{\partial^{t+\Delta t}\mathbf{u}}{\partial\xi^i}\cdot\frac{\partial\delta^{t+\Delta t}\mathbf{u}}{\partial\xi^j}\right)$$

$$=\frac{1}{2}\left(\begin{array}{l}\dfrac{\partial^0\mathbf{X}}{\partial\xi^i}\cdot\dfrac{\partial\delta\left({}^{t}\mathbf{N}^{t+\Delta t}\mathbf{u}_n+{}^{0}\tilde{\mathbf{N}}\Delta^0\tilde{\mathbf{X}}_n\right)}{\partial\xi^j}+\dfrac{\partial\delta\left({}^{t}\mathbf{N}^{t+\Delta t}\mathbf{u}_n+{}^{0}\tilde{\mathbf{N}}\Delta^0\tilde{\mathbf{X}}_n\right)}{\partial\xi^i}\cdot\dfrac{\partial^0\mathbf{X}}{\partial\xi^j}\\ +\dfrac{\partial\left({}^{t}\mathbf{N}^{t+\Delta t}\mathbf{u}_n+{}^{0}\tilde{\mathbf{N}}\Delta^0\tilde{\mathbf{X}}_n\right)}{\partial\xi^i}\cdot\dfrac{\partial^{t+\Delta t}\mathbf{u}}{\partial\xi^j}+\dfrac{\partial^{t+\Delta t}\mathbf{u}}{\partial\xi^i}\cdot\dfrac{\partial\left({}^{t}\mathbf{N}^{t+\Delta t}\mathbf{u}_n+{}^{0}\tilde{\mathbf{N}}\Delta^0\tilde{\mathbf{X}}_n\right)}{\partial\xi^j}\end{array}\right)$$

$$=\frac{1}{2}\left(\frac{\partial^0\mathbf{X}}{\partial\xi^i}\cdot\frac{\partial\delta^t\mathbf{N}^{t+\Delta t}\mathbf{u}_n}{\partial\xi^j}+\frac{\partial\delta^t\mathbf{N}^{t+\Delta t}\mathbf{u}_n}{\partial\xi^i}\cdot\frac{\partial^0\mathbf{X}}{\partial\xi^j}+\frac{\partial^t\mathbf{N}^{t+\Delta t}\mathbf{u}_n}{\partial\xi^i}\cdot\frac{\partial^{t+\Delta t}\mathbf{u}}{\partial\xi^j}+\frac{\partial^{t+\Delta t}\mathbf{u}}{\partial\xi^i}\cdot\frac{\partial^t\mathbf{N}^{t+\Delta t}\mathbf{u}_n}{\partial\xi^j}\right)$$

$$=\left[\delta^{t+\Delta t}\mathbf{u}_n\right]^T\frac{1}{2}\left\{\left[\frac{\partial^0\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^t\mathbf{N}}{\partial\xi^j}\right]+\left[\frac{\partial^t\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^0\mathbf{N}}{\partial\xi^j}\right]\right\}\left[{}^{0}\mathbf{X}_n\right]$$

$$+\left[\delta^{t+\Delta t}\mathbf{u}_n\right]^T\frac{1}{2}\left\{\left[\frac{\partial^t\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^0\mathbf{N}}{\partial\xi^j}\right]+\left[\frac{\partial^0\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^t\mathbf{N}}{\partial\xi^j}\right]\right\}\left[{}^{t}\mathbf{x}_n-{}^{0}\mathbf{X}_n\right]$$

$$=\left[\delta^{t+\Delta t}\mathbf{u}_n\right]^T\underbrace{\frac{1}{2}\left\{\left[\frac{\partial^0\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^t\mathbf{N}}{\partial\xi^j}\right]+\left[\frac{\partial^t\mathbf{N}}{\partial\xi^i}\right]^T\left[\frac{\partial^0\mathbf{N}}{\partial\xi^j}\right]\right\}}_{[\mathbf{a}]_{ij}}\left[{}^{t}\mathbf{x}_n\right]$$

$$\left[\delta^{t+\Delta t}{}_{0}\mathbf{E}_L\right]_{ij}=\frac{1}{2}\left(\frac{\partial\delta\left({}^{t}\mathbf{N}^{t+\Delta t}\mathbf{u}_n+{}^{0}\tilde{\mathbf{N}}\Delta^0\tilde{\mathbf{X}}_n\right)}{\partial\xi^i}\cdot\frac{\partial\Delta_0\mathbf{u}}{\partial\xi^j}+\frac{\partial\Delta_0\mathbf{u}}{\partial\xi^i}\cdot\frac{\partial\delta\left({}^{t}\mathbf{N}^{t+\Delta t}\mathbf{u}_n+{}^{0}\tilde{\mathbf{N}}\Delta^0\tilde{\mathbf{X}}_n\right)}{\partial\xi^j}\right)$$

$$=\frac{1}{2}\left(\frac{\partial\delta^t\mathbf{N}^{t+\Delta t}\mathbf{u}_n}{\partial\xi^i}\cdot\frac{\partial\Delta_0\mathbf{u}}{\partial\xi^j}+\frac{\partial\Delta_0\mathbf{u}}{\partial\xi^i}\cdot\frac{\partial\delta^t\mathbf{N}^{t+\Delta t}\mathbf{u}_n}{\partial\xi^j}\right)$$

$$=\left[\delta^{t+\Delta t}\mathbf{u}_n\right]^T\underbrace{\frac{1}{2}\left(\left[\frac{\partial^t\mathbf{N}}{\partial\xi^i}\right]^T\cdot\left[\frac{\partial^t\mathbf{N}}{\partial\xi^j}\right]+\left[\frac{\partial^t\mathbf{N}}{\partial\xi^i}\right]^T\cdot\left[\frac{\partial^t\mathbf{N}}{\partial\xi^j}\right]\right)}_{[\mathbf{c}]_{ij}}\left[\Delta^t\mathbf{u}_n\right]$$

다시 가상일 항으로 돌아와서 위에 구한 Green-Lagrange strain을 대입하면

$$\int_{V_0}\underbrace{\left[\delta^{t+\Delta t}{}_{0}\mathbf{E}_C\right]_{ij}\left[{}^{t+\Delta t}{}_{0}\mathbf{S}_0\right]^{ij}}_{\text{constant term}}dV_0+\int_{V_0}\underbrace{\left[\delta^{t+\Delta t}{}_{0}\mathbf{E}_C\right]_{ij}\left[{}^{t+\Delta t}{}_{0}\mathbf{S}_C\right]^{ij}+\left[\delta^{t+\Delta t}{}_{0}\mathbf{E}_L\right]_{ij}\left[{}^{t+\Delta t}{}_{0}\mathbf{S}_0\right]^{ij}}_{\text{linear term}}dV_0$$

$$\left(\begin{array}{l}\left[\delta^{t+\Delta t}{}_{0}\mathbf{E}_C\right]_{ij}\left[{}^{t+\Delta t}{}_{0}\mathbf{S}_0\right]^{ij}=\left[\delta^{t+\Delta t}\mathbf{u}_n\right]^T[\mathbf{a}]_{ij}\left[{}^{t}\mathbf{x}_n\right]\left[{}^{t+\Delta t}{}_{0}\mathbf{C}\right]^{ijkl}\dfrac{1}{2}\left[{}^{t}\mathbf{x}_n+{}^{0}\mathbf{X}_n\right]^T[\mathbf{e}]_{kl}\left[{}^{t}\mathbf{x}_n-{}^{0}\mathbf{X}_n\right]\\ \left[\delta^{t+\Delta t}{}_{0}\mathbf{E}_C\right]_{ij}\left[{}^{t+\Delta t}{}_{0}\mathbf{S}_C\right]^{ij}=\left[\delta^{t+\Delta t}\mathbf{u}_n\right]^T[\mathbf{a}]_{ij}\left[{}^{t}\mathbf{x}_n\right]\left[{}^{t+\Delta t}{}_{0}\mathbf{C}\right]^{ijkl}\left[{}^{t}\mathbf{x}_n\right]^T[\mathbf{a}]_{kl}\left[\Delta^t\mathbf{u}_n\right]\\ \left[\delta^{t+\Delta t}{}_{0}\mathbf{E}_L\right]_{ij}\left[{}^{t+\Delta t}{}_{0}\mathbf{S}_0\right]^{ij}=\left[\delta^{t+\Delta t}\mathbf{u}_n\right]^T[\mathbf{c}]_{ij}\left[\Delta^t\mathbf{u}_n\right]\left[{}^{t+\Delta t}{}_{0}\mathbf{C}\right]^{ijkl}\dfrac{1}{2}\left[{}^{t}\mathbf{x}_n+{}^{0}\mathbf{X}_n\right]^T[\mathbf{e}]_{kl}\left[{}^{t}\mathbf{x}_n-{}^{0}\mathbf{X}_n\right]\end{array}\right)$$

$$=\left[\delta^{t+\Delta t}\mathbf{u}_n\right]^T\left\{\begin{array}{l}\left(\int_{V_0}[\mathbf{a}]_{ij}\left[{}^{t+\Delta t}{}_{0}\mathbf{S}_0\right]^{ij}dV_0\right)\left[{}^{t}\mathbf{x}_n\right]\\ \quad+\left(\int_{V_0}[\mathbf{a}]_{ij}\left[{}^{t}\mathbf{x}_n\right]\left[{}^{t+\Delta t}{}_{0}\mathbf{C}\right]^{ijkl}\left[{}^{t}\mathbf{x}_n\right]^T[\mathbf{a}]_{kl}+[\mathbf{c}]_{ij}\left[{}^{t+\Delta t}{}_{0}\mathbf{S}_0\right]^{ij}dV_0\right)\left[\Delta^t\mathbf{u}_n\right]\end{array}\right\}$$

와 같다. 이제 우변의 항들을 정리해보면 아래와 같다. 여기서 body force에 대한 영향은 무시한다.

$$\int_{\partial V_{0m}} \delta^{t+\Delta t}\mathbf{u}^{t+\Delta t}{}_0\mathbf{F}^{t+\Delta t}{}_0\mathbf{S}^{t+\Delta t}\tilde{\mathbf{n}}dA_0 = \int_{\partial V_{0m}} \delta\left({}^t\mathbf{N}^{t+\Delta t}\mathbf{u}_n + {}^0\tilde{\mathbf{N}}\Delta^0\tilde{\mathbf{X}}_n\right)^{t+\Delta t}{}_0\mathbf{F}^{t+\Delta t}{}_0\mathbf{S}^{t+\Delta t}\tilde{\mathbf{n}}dA_0$$

$$= \int_{\partial V_{0m}} \delta\left({}^t\mathbf{N}^{t+\Delta t}\mathbf{u}_n\right)^{t+\Delta t}{}_0\mathbf{F}^{t+\Delta t}{}_0\mathbf{S}^{t+\Delta t}\tilde{\mathbf{n}}dA_0$$

$$= \left[\delta^{t+\Delta t}\mathbf{u}_n\right]^T \int_{\partial V_{0m}} \left[{}^t\mathbf{N}\right]^{T\,t+\Delta t}{}_0\mathbf{F}^{t+\Delta t}{}_0\mathbf{S}^{t+\Delta t}\tilde{\mathbf{n}}dA_0$$

$$= \left[\delta^{t+\Delta t}\mathbf{u}_n\right]^T \int_{\partial V_{0m}} \left[{}^t\mathbf{N}\right]^T \left[\mathbf{t}\right] dA_0$$

따라서 가상변위를 지워 모든 식을 정리하면

$$\int_{V_0} \delta^{t+\Delta t}\mathbf{u}\cdot\rho_0{}^{t+\Delta t}\ddot{\mathbf{u}}dV_0 + \int_{V_0} \delta^{t+\Delta t}{}_0\mathbf{E} : {}^{t+\Delta t}_0\mathbf{S}dV_0 = \int_{\partial V_{0m}} \delta^{t+\Delta t}\mathbf{u}^{t+\Delta t}{}_0\mathbf{F}^{t+\Delta t}{}_0\mathbf{S}^{t+\Delta t}\tilde{\mathbf{n}}dA_0 + \int_{V_0} \delta^{t+\Delta t}\mathbf{u}\rho_0{}^{t+\Delta t}\mathbf{f}dV_0$$

$$\Rightarrow \underbrace{\int_{V_0} \rho\left[{}^t\mathbf{N}\right]^T\left[{}^t\mathbf{N}\right]JdV_0}_{\mathbf{M}}\,{}^{t+\Delta t}\ddot{\mathbf{u}} + \underbrace{\int_{V_0}\left[\mathbf{a}\right]_{ij}\left[{}^{t+\Delta t}{}_0\mathbf{S}_0\right]^{ij}dV_0\,{}^t\mathbf{x}}_{\mathbf{f}_{\text{int}}}$$

$$+ \underbrace{\int_{V_0}\left[\mathbf{a}\right]_{ij}\left[{}^t\mathbf{x}_n\right]\left[{}^{t+\Delta t}{}_0\mathbf{C}\right]^{ijkl}\left[{}^t\mathbf{x}_n\right]^T\left[\mathbf{a}\right]_{kl} + \left[\mathbf{c}\right]_{ij}\left[{}^{t+\Delta t}{}_0\mathbf{S}_0\right]^{ij}dV_0}_{\mathbf{K}_t}\,\Delta^t\mathbf{u} = \underbrace{\int_{\partial V_{0m}}\left[{}^t\mathbf{N}\right]^T\left[\mathbf{t}\right]dA_0}_{\mathbf{P}_{dist}} + \mathbf{P}_{con}$$

$$\Rightarrow \mathbf{M}\,{}^{t+\Delta t}\ddot{\mathbf{u}} + \mathbf{K}_t\Delta^t\mathbf{u} = \mathbf{P}_{dist} + \mathbf{P}_{con} - \mathbf{f}_{\text{int}}$$

와 같이 정리할 수 있다. 여기서 $\mathbf{M}$은 mass matrix, $\mathbf{K}_t$는 tangent stiffness matrix, $\mathbf{P}_{dist}$는 분포하중에 의한 힘, $\mathbf{P}_{con}$는 집중하중, $\mathbf{f}_{\text{int}}$ 는 변형에 의한 힘을 나타낸다.

4. Constitutive matrix

Plane stress 가정을 사용하는 구성행렬은 다음과 같다.

$$\left[{}^{t+\Delta t}{}_0\mathbf{C}\right]_{x^1x^2x^3} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 & 0 & 0 & 0 \\ n & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \kappa\frac{1-\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \kappa\frac{1-\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa\frac{1-\nu}{2} \end{bmatrix}$$

위의 구성 행렬은 local Cartesian coordinate에서 정의되었기 때문에 transformation matrix를 이용하여 natural coordinate로 바꾸어 줄 수 있다.

$$\left[{}^{t+\Delta t}{}_0\mathbf{C}\right]_{\xi^1\xi^2\xi^3} = \left[{}^{t+\Delta t}{}_0\mathbf{T}\right]^T\left[{}^{t+\Delta t}{}_0\mathbf{C}\right]_{x^1x^2x^3}\left[{}^{t+\Delta t}{}_0\mathbf{T}\right]$$

이 때 전단보정계수 κ는 $\frac{5}{6}$를 사용하였다.

$$E_{kl} = \tilde{E}_{mn} \underbrace{\left(\mathbf{E}_k \cdot \mathbf{G}^m\right)\left(\mathbf{E}_l \cdot \mathbf{G}^n\right)}_{\equiv \mathbf{T}} = \tilde{E}_{mn} \left(\mathbf{E}_k \cdot \frac{\partial X^a}{\partial \xi^m} \mathbf{E}_a \right)\left(\mathbf{E}_l \cdot \frac{\partial X^b}{\partial \xi^n} \mathbf{E}_b \right) = \frac{\partial X^k}{\partial \xi^m} \frac{\partial X^l}{\partial \xi^n}$$

$$[\mathbf{T}] = \begin{bmatrix}
\frac{\partial X^1}{\partial \xi^1}\frac{\partial X^1}{\partial \xi^1} & \frac{\partial X^1}{\partial \xi^2}\frac{\partial X^1}{\partial \xi^2} & \frac{\partial X^1}{\partial \xi^3}\frac{\partial X^1}{\partial \xi^3} & \frac{\partial X^1}{\partial \xi^2}\frac{\partial X^1}{\partial \xi^3} & \frac{\partial X^1}{\partial \xi^1}\frac{\partial X^1}{\partial \xi^3} & \frac{\partial X^1}{\partial \xi^1}\frac{\partial X^1}{\partial \xi^2} \\
\frac{\partial X^2}{\partial \xi^1}\frac{\partial X^2}{\partial \xi^1} & \frac{\partial X^2}{\partial \xi^2}\frac{\partial X^2}{\partial \xi^2} & \frac{\partial X^2}{\partial \xi^3}\frac{\partial X^2}{\partial \xi^3} & \frac{\partial X^2}{\partial \xi^2}\frac{\partial X^2}{\partial \xi^3} & \frac{\partial X^2}{\partial \xi^1}\frac{\partial X^2}{\partial \xi^3} & \frac{\partial X^2}{\partial \xi^1}\frac{\partial X^2}{\partial \xi^2} \\
\frac{\partial X^3}{\partial \xi^1}\frac{\partial X^3}{\partial \xi^1} & \frac{\partial X^3}{\partial \xi^2}\frac{\partial X^3}{\partial \xi^2} & \frac{\partial X^3}{\partial \xi^3}\frac{\partial X^3}{\partial \xi^3} & \frac{\partial X^3}{\partial \xi^2}\frac{\partial X^3}{\partial \xi^3} & \frac{\partial X^3}{\partial \xi^1}\frac{\partial X^3}{\partial \xi^3} & \frac{\partial X^3}{\partial \xi^1}\frac{\partial X^3}{\partial \xi^2} \\
2\frac{\partial X^2}{\partial \xi^1}\frac{\partial X^3}{\partial \xi^1} & 2\frac{\partial X^2}{\partial \xi^2}\frac{\partial X^3}{\partial \xi^2} & 2\frac{\partial X^2}{\partial \xi^3}\frac{\partial X^3}{\partial \xi^3} & \frac{\partial X^2}{\partial \xi^2}\frac{\partial X^3}{\partial \xi^3}+\frac{\partial X^2}{\partial \xi^3}\frac{\partial X^3}{\partial \xi^2} & \frac{\partial X^2}{\partial \xi^1}\frac{\partial X^3}{\partial \xi^3}+\frac{\partial X^2}{\partial \xi^3}\frac{\partial X^3}{\partial \xi^1} & \frac{\partial X^2}{\partial \xi^1}\frac{\partial X^3}{\partial \xi^2}+\frac{\partial X^2}{\partial \xi^2}\frac{\partial X^3}{\partial \xi^1} \\
2\frac{\partial X^1}{\partial \xi^1}\frac{\partial X^3}{\partial \xi^1} & 2\frac{\partial X^1}{\partial \xi^2}\frac{\partial X^3}{\partial \xi^2} & 2\frac{\partial X^1}{\partial \xi^3}\frac{\partial X^3}{\partial \xi^3} & \frac{\partial X^1}{\partial \xi^2}\frac{\partial X^3}{\partial \xi^3}+\frac{\partial X^1}{\partial \xi^3}\frac{\partial X^3}{\partial \xi^2} & \frac{\partial X^1}{\partial \xi^1}\frac{\partial X^3}{\partial \xi^3}+\frac{\partial X^1}{\partial \xi^3}\frac{\partial X^3}{\partial \xi^1} & \frac{\partial X^1}{\partial \xi^1}\frac{\partial X^3}{\partial \xi^2}+\frac{\partial X^1}{\partial \xi^2}\frac{\partial X^3}{\partial \xi^1} \\
2\frac{\partial X^1}{\partial \xi^1}\frac{\partial X^2}{\partial \xi^1} & 2\frac{\partial X^1}{\partial \xi^2}\frac{\partial X^2}{\partial \xi^2} & 2\frac{\partial X^1}{\partial \xi^3}\frac{\partial X^2}{\partial \xi^3} & \frac{\partial X^1}{\partial \xi^2}\frac{\partial X^2}{\partial \xi^3}+\frac{\partial X^1}{\partial \xi^3}\frac{\partial X^2}{\partial \xi^2} & \frac{\partial X^1}{\partial \xi^1}\frac{\partial X^2}{\partial \xi^3}+\frac{\partial X^1}{\partial \xi^3}\frac{\partial X^2}{\partial \xi^1} & \frac{\partial X^1}{\partial \xi^1}\frac{\partial X^2}{\partial \xi^2}+\frac{\partial X^1}{\partial \xi^2}\frac{\partial X^2}{\partial \xi^1}
\end{bmatrix}$$

5. Nonlinear Newmark- β integration method

먼저 물체의 비선형 운동방정식은 다음과 같다.

$$\mathbf{M}\ddot{\mathbf{u}} + \mathbf{C}(\dot{\mathbf{u}})\dot{\mathbf{u}} + \mathbf{K}(\mathbf{u})\mathbf{u} = \mathbf{P}$$

여기서 밀도는 시간이나 변위에 따라 변화하지 않는다고 가정하면 $\mathbf{M}$은 항상 일정하다. 또한 구조물의 동적 문제이기 때문에 $\mathbf{C}$는 없다고 생각 할 수 있다. $n+1$시간에서 평형방정식을 생각하면

$$\mathbf{M}\ddot{\mathbf{u}}_{n+1} + \mathbf{K}(\mathbf{u}_{n+1})\mathbf{u}_{n+1} = \mathbf{P}_{n+1}$$

와 같다. 운동방정식이 비선형이기 때문에 $n+1$시간에서 평형을 만족하는 변위와 가속도를 계산하기 위해서 반복 계산이 필요하다. 따라서 Newton-Raphson method를 사용하여 반복계산을 수행하였다. $k+1$번째 반복에서 평형이 이루어졌다면 식은 다음과 같다.

$$\mathbf{M}\ddot{\mathbf{u}}_{n+1}^{k+1} + \mathbf{K}\left(\mathbf{u}_{n+1}^{k+1}\right)\mathbf{u}_{n+1}^{k+1} = \mathbf{P}_{n+1}^{k+1}$$

위 식을 정리하면

$$\mathbf{M}\ddot{\mathbf{u}}_{n+1}^{k+1} + \mathbf{K}\left(\mathbf{u}_{n+1}^{k+1}\right)\mathbf{u}_{n+1}^{k+1} - \mathbf{P}_{n+1}^{k+1} = 0 = \mathbf{R}_{n+1}^{k+1}$$

와 같고 Taylor series expansion을 통해 선형화 시키면

$$\mathbf{R}_{n+1}^{k+1} = \mathbf{R}_{n+1}^{k} + \frac{\partial \mathbf{R}_{n+1}^{k}}{\partial \mathbf{u}_{n+1}^{k}} \Delta \mathbf{u}_{n+1}^{k} + \frac{\partial \mathbf{R}_{n+1}^{k}}{\partial \dot{\mathbf{u}}_{n+1}^{k}} \Delta \dot{\mathbf{u}}_{n+1}^{k} + \frac{\partial \mathbf{R}_{n+1}^{k}}{\partial \ddot{\mathbf{u}}_{n+1}^{k}} \Delta \ddot{\mathbf{u}}_{n+1}^{k}$$

와 같다. 이를 풀어 쓰면

$$0=\mathbf{P}_{n+1}^k+\underbrace{\frac{\partial \mathbf{P}_{n+1}^k}{\partial \mathbf{u}_{n+1}^k}}_{\mathbf{P}_t}\Delta\mathbf{u}_{n+1}^k-\left\{\mathbf{M}\ddot{\mathbf{u}}_{n+1}^k+\underbrace{\mathbf{K}\left(\mathbf{u}_{n+1}^k\right)\mathbf{u}_{n+1}^k}_{\mathbf{f}_{\text{int}}\left(\mathbf{u}_{n+1}^k\right)}\right\}-\left\{\mathbf{M}\Delta\ddot{\mathbf{u}}_{n+1}^k+\underbrace{\frac{\partial\left(\mathbf{K}\left(\mathbf{u}_{n+1}^k\right)\mathbf{u}_{n+1}^k\right)}{\partial \mathbf{u}_{n+1}^k}}_{\mathbf{K}_t}\Delta\mathbf{u}_{n+1}^k\right\}$$

$$\Rightarrow \mathbf{M}\Delta\ddot{\mathbf{u}}_{n+1}^k+\mathbf{K}_t\Delta\mathbf{u}_{n+1}^k-\mathbf{P}_t\Delta\mathbf{u}_{n+1}^k=\mathbf{P}_{n+1}^k-\left\{\mathbf{M}\ddot{\mathbf{u}}_{n+1}^k+\mathbf{f}_{\text{int}}\left(\mathbf{u}_{n+1}^k\right)\right\}$$

와 같다. Newmark-β method를 적용하면 $n+1$시간에서 변위와 속도를 구할 수 있다.

$$\mathbf{u}_{n+1}=\mathbf{u}_n+h\dot{\mathbf{u}}_n+h^2\left(\frac{1}{2}-\beta\right)\ddot{\mathbf{u}}_n+h^2\beta\ddot{\mathbf{u}}_{n+1}=\mathbf{u}_n+h\dot{\mathbf{u}}_n+\frac{h^2}{2}\ddot{\mathbf{u}}_n-h^2\beta\ddot{\mathbf{u}}_n+h^2\beta\ddot{\mathbf{u}}_{n+1}$$
$$\dot{\mathbf{u}}_{n+1}=\dot{\mathbf{u}}_n+h\left(1-\gamma\right)\ddot{\mathbf{u}}_n+h\gamma\ddot{\mathbf{u}}_{n+1}=\dot{\mathbf{u}}_n+h\ddot{\mathbf{u}}_n+\gamma h\ddot{\mathbf{u}}_{n+1}-\gamma h\ddot{\mathbf{u}}_n$$

위의 식을 가속도와 속도로 나타내면

$$h^2\beta\ddot{\mathbf{u}}_{n+1}=\mathbf{u}_{n+1}-\mathbf{u}_n-h\dot{\mathbf{u}}_n-\frac{h^2}{2}\ddot{\mathbf{u}}_n+h^2\beta\ddot{\mathbf{u}}_n$$
$$\dot{\mathbf{u}}_{n+1}=\dot{\mathbf{u}}_n+h\ddot{\mathbf{u}}_n+\gamma h\ddot{\mathbf{u}}_{n+1}-\gamma h\ddot{\mathbf{u}}_n$$

여기서 마찬가지로 $k+1$ 반복에서 평형을 이룬다면

$$h^2\beta\ddot{\mathbf{u}}_{n+1}^{k+1}=\mathbf{u}_{n+1}^{k+1}-\mathbf{u}_n-h\dot{\mathbf{u}}_n-\frac{h^2}{2}\ddot{\mathbf{u}}_n+h^2\beta\ddot{\mathbf{u}}_n$$
$$\dot{\mathbf{u}}_{n+1}^{k+1}=\dot{\mathbf{u}}_n+h\ddot{\mathbf{u}}_n+\gamma h\ddot{\mathbf{u}}_{n+1}^{k+1}-\gamma h\ddot{\mathbf{u}}_n$$

와 같고 반복에 대한 항을 선형화 시키면

$$h^2\beta\ddot{\mathbf{u}}_{n+1}^k+h^2\beta\Delta\ddot{\mathbf{u}}_{n+1}^k=\mathbf{u}_{n+1}^k+\Delta\mathbf{u}_{n+1}^k-\mathbf{u}_n-h\dot{\mathbf{u}}_n-\frac{h^2}{2}\ddot{\mathbf{u}}_n+h^2\beta\ddot{\mathbf{u}}_n$$
$$\dot{\mathbf{u}}_{n+1}^k+\Delta\dot{\mathbf{u}}_{n+1}^k=\dot{\mathbf{u}}_n+h\ddot{\mathbf{u}}_n+\gamma h\ddot{\mathbf{u}}_{n+1}^k+\gamma h\Delta\ddot{\mathbf{u}}_{n+1}^k-\gamma h\ddot{\mathbf{u}}_n$$

위 식을 다음과 같이 k 번째 반복의 가속도와 속도, k 번째 반복의 미소 가속도와 미소 속도 항으로 분리 할 수 있다.

$$\ddot{\mathbf{u}}_{n+1}^k=\frac{1}{h^2\beta}\mathbf{u}_{n+1}^k-\frac{1}{h^2\beta}\mathbf{u}_n-\frac{1}{h\beta}\dot{\mathbf{u}}_n-\frac{1}{2\beta}\ddot{\mathbf{u}}_n+\ddot{\mathbf{u}}_n$$
$$\dot{\mathbf{u}}_{n+1}^k=\dot{\mathbf{u}}_n+h\ddot{\mathbf{u}}_n+\gamma h\ddot{\mathbf{u}}_{n+1}^k-\gamma h\ddot{\mathbf{u}}_n=\frac{\gamma}{h\beta}\mathbf{u}_{n+1}^k-\frac{\gamma}{h\beta}\mathbf{u}_n+\left(1-\frac{\gamma}{\beta}\right)\dot{\mathbf{u}}_n+h\left(1-\frac{\gamma}{2\beta}\right)\ddot{\mathbf{u}}_n$$
$$\Delta\ddot{\mathbf{u}}_{n+1}^k=\frac{1}{h^2\beta}\Delta\mathbf{u}_{n+1}^k$$
$$\Delta\dot{\mathbf{u}}_{n+1}^k=\gamma h\Delta\ddot{\mathbf{u}}_{n+1}^k=\frac{\gamma}{h\beta}\Delta\mathbf{u}_{n+1}^k$$

위의 미소 가속도, 미소 속도를 대입하면

$$\mathbf{M}\Delta\ddot{\mathbf{u}}_{n+1}^{k} + \mathbf{K}_t \Delta\mathbf{u}_{n+1}^{k} - \mathbf{P}_t \Delta\mathbf{u}_{n+1}^{k} = \mathbf{P}_{n+1}^{k} - \left\{ \mathbf{M}\ddot{\mathbf{u}}_{n+1}^{k} + \mathbf{f}_{\text{int}}\left(\mathbf{u}_{n+1}^{k}\right) \right\}$$

$$\Rightarrow \left[\frac{1}{h^2\beta}\mathbf{M} + \mathbf{K}_t\left(\mathbf{u}_{n+1}^{k}\right) - \mathbf{P}_t\left(\mathbf{u}_{n+1}^{k}\right) \right] \Delta\mathbf{u}_{n+1}^{k} = \underbrace{\mathbf{P}_{n+1}^{k} - \left\{ \mathbf{M}\ddot{\mathbf{u}}_{n+1}^{k} + \mathbf{f}_{\text{int}}\left(\mathbf{u}_{n+1}^{k}\right) \right\}}_{\mathbf{R}\left(\mathbf{u}_{n+1}^{k}\right)}$$

여기서 $\mathbf{f}_{\text{int}}\left(\mathbf{u}_{n+1}^{k}\right)$와 $\mathbf{K}_t\left(\mathbf{u}_{n+1}^{k}\right)$는 $\mathbf{u}_{n+1}^{k}$의 함수이기 때문에 반복이 수행될 때마다 다시 계산해 주어야 한다. 이후 다음 반복에 대한 변위, 속도, 가속도는 다음과 같다.

$$\mathbf{u}_{n+1}^{k+1} = \mathbf{u}_{n+1}^{k} + \Delta\mathbf{u}_{n+1}^{k}$$

$$\dot{\mathbf{u}}_{n+1}^{k+1} = \frac{\gamma}{h\beta}\mathbf{u}_{n+1}^{k+1} - \left(\frac{\gamma}{h\beta}\mathbf{u}_n - \left(1 - \frac{\gamma}{\beta}\right)\dot{\mathbf{u}}_n - h\left(1 - \frac{\gamma}{2\beta}\right)\ddot{\mathbf{u}}_n \right)$$

$$\ddot{\mathbf{u}}_{n+1}^{k+1} = \frac{1}{h^2\beta}\mathbf{u}_{n+1}^{k+1} - \left(\frac{1}{h^2\beta}\mathbf{u}_n + \frac{1}{h\beta}\dot{\mathbf{u}}_n + \frac{1}{2\beta}\ddot{\mathbf{u}}_n - \ddot{\mathbf{u}}_n \right)$$